Effective Field Theories for Heavy Quarks

Part I: Generalities

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Overview

- Part I:
  Generalities of Effective Field Therories
- Part II:
  Heavy Quark Effective Theory
- Part III:
  Heavy Quark expansion /
  Soft Collinear Effective Theory

Literature for Part I

(a personal selection)

- I. Rothstein, hep-ph/0308266
- A. Manohar, hep-ph/9508245, hep-ph/9606222
- C. Burgess, hep-ph/9812470, hep-th/0701053
- G. P. Lepage, hep-ph/0506330 (on renormalization)
- A. Buras, G. Buchalla, M. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996) [arXiv:hep-ph/9512380]. (weak interaction)

Contents Part I

1. Introduction
   - Generalities
   - Disparate Mass Scales
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   - A QFT Nutshell
   - Toy Example
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3. Renormalization
   - Why Renormalization?
   - Renormalization Group
   - Leading and Sub-Leading Logs

Preface

Effective Theories are everywhere:

- Classical Mechanics emerges from Quantum Mechanics in the limit $\hbar \to 0$
- Classical Mechanics emerges from Relativistic Mechanics in the limit $c \to \infty$
- Newtonian Mechanics emerges from General Relativity in the limit $c \to \infty$
- ....
- Fermi Weak Interactions emerges from the Standard Model in the limit $M_W \to \infty$
- HQET emerges from QCD in the limit $m_Q \to \infty$

Disparate Mass Scales

- Intuitively: Very heavy particles (“degrees of freedom”) cannot play a role at low energies (“at low scales”)
- In a field theory language: Formulate a theory in terms of the light particles (“light fields”) ONLY!
- Start from a “full theory” (with light and heavy degrees of freedom) and construct an “effective field theory” in terms of light fields
- “full theory” can be an effective theory as well
- ... tower of effective theories as the energy scale changes

- Relevant example: Weak decays of heavy quarks:
  Very different mass scales are involved:
  - $\Lambda_{\mathrm{QCD}} \sim 200$ MeV: Scale of strong interactions
  - $m_c \sim 1.5$ GeV: Charm Quark Mass
  - $m_b \sim 4.5$ GeV: Bottom Quark Mass
  - $m_t \sim 175$ GeV and $M_W \sim 81$ GeV:
    Top Quark Mass and Weak Boson Mass
  - Λ_{NP} Scale of "new physics"
- At low scales the high mass particles / high energy degrees of freedom are irrelevant.
- Construct an "effective field theory" where the massive / energetic degrees of freedom are removed ("integrated out")

QFT Preliminaries

- Central Quantities: (n-point) Greens Functions

$$G^{(n)}(x_1, x_2, ..., x_n) = \langle 0 | T[\phi(x_1)\phi(x_2)...\phi(x_n)] | 0 \rangle$$

- May be written as a "Functional Integral"

$$G^{(n)}(x_i) = \int [d\phi]\, \phi(x_1)\phi(x_2)...\phi(x_n) \exp\left(-i \int d^4x\, \mathcal{L}(\phi)\right)$$

- Functional integral = Short-hand notation for all Feynman Diagramms contributing to the particular Greens function = Perturbative expansion

- Generating Functional for the Greens functions:

$$Z[j] = \int [d\phi] \exp\left(-i \int d^4y \, [\mathcal{L}(\phi) + j(y)\phi(y)]\right)$$

- Obtain all Greens functions from (functional) differentiation:

$$G^{(n)}(x_1, x_2, ..., x_n) = (i)^n \frac{\partial}{\partial j(x_1)} \frac{\partial}{\partial j(x_2)} \cdots \frac{\partial}{\partial j(x_n)} Z[j]\Big|_{j\equiv 0}$$

- $Z[j]$: Short-hand Notation for a complete QFT
- Generalization to more than one field ϕ obvious

Integrating out heavy degrees of freedom

- ϕ: light fields, Φ: heavy fields with mass Λ
- Generating functional as a functional integral
  Integration over the heavy degrees of freedom

$$Z[j] = \int [d\phi][d\Phi] \exp\left(\int d^4x \, [\mathcal{L}(\phi, \Phi) + j\phi] \right)$$

$$= \int [d\phi] \exp\left(\int d^4x \, [\mathcal{L}_{\rm eff}(\phi) + j\phi] \right) \quad \text{with}$$

$$\exp\left(\int d^4x \, \mathcal{L}_{\rm eff}(\phi) \right) = \int [d\Phi] \exp\left(\int d^4x \, \mathcal{L}(\phi, \Phi) \right)$$

- For length scales $x \gg 1/\Lambda$: local effective Lagrangian
- Technically: (Operator Product) Expansion in inverse powers of Λ

$$\mathcal{L}_{\rm eff}(\phi) = \mathcal{L}_{\rm eff}^{(4)}(\phi) + \frac{1}{\Lambda}\mathcal{L}_{\rm eff}^{(5)}(\phi) + \frac{1}{\Lambda^2}\mathcal{L}_{\rm eff}^{(6)}(\phi) + \cdots$$

- $\mathcal{L}_{\rm eff}$ is in general non-renormalizable, but ...
- $\mathcal{L}_{\rm eff}^{(4)}$ is the renormalizable piece
- For a fixed order in $1/\Lambda$: Only a finite number of insertions of $\mathcal{L}_{\rm eff}^{(4)}$ is needed!
- $\rightarrow$ can be renormalized
- Renormalizability is not an issue here

Toy Example

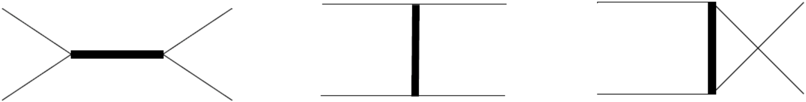
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2$$
$$-\frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_3}{4!}\Phi^4 - \frac{\lambda_1}{2}\Phi\phi^2 - \frac{\lambda_2}{4}\Phi^2\phi^2$$

- Integrate out Φ, assuming that $M \gg m$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$
$$+\text{Terms originating from the presence of } \Phi$$

- How to construct these terms?

... Calculate Feynman Diagrams: Four Point Function:



$$= \frac{-i\lambda_1^2}{s - M^2} + \frac{-i\lambda_1^2}{t - M^2} + \frac{-i\lambda_1^2}{u - M^2}$$

- For small energies: $s, t, u \ll M^2$

$$\text{Diagrams} = (-i)\frac{-3\lambda_1^2}{M^2} + \mathcal{O}\left(\frac{s}{M^2}, \frac{t}{M^2}, \frac{u}{M^2}\right)$$

- To leading order in $(s, t, u)/M^2$: Modification of λ_0:

$$-\frac{\lambda_0}{4!}\phi^4 \to -C_4^{(1,0)}\frac{1}{4!}\phi^4 = -\frac{1}{4!}\left(\lambda_0 - \frac{-3\lambda_1^2}{M^2}\right)\phi^4$$

- $C_4^{(1,0)}$: Wilson Coefficient
- What, if we increase the energy? Take the next term

$$\text{Diagrams} = (-i)\frac{-3\lambda_1^2}{M^2} + (-i)\frac{\lambda_1^2}{M^2}\left(\frac{s}{M^2} + \frac{t}{M^2} + \frac{u}{M^2}\right) + \dots$$

- Can be incorporated as an additional term in $\mathcal{L}_{\text{eff}}$:

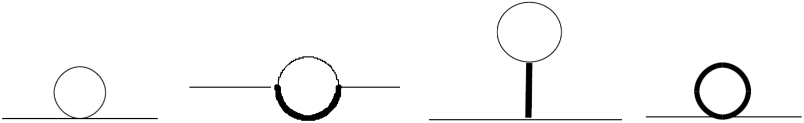
$$-\frac{1}{4!}\frac{C_4^{(1,2)}}{M^2}\left[(\partial_\mu\phi)(\partial^\mu\phi)\phi^2 \right.$$
$$\left. + \text{ all other operators with four } \phi\text{'s and two derivatives}\right]$$

Remarks:

- At very small energies the observers remains completely ignorant about the UV physics = the heavy field Φ
- Any sensitivity to UV physics is encoded in "higher dimensional operators" = operators with more fields ϕ and more derivatives.
- Systematic expansion in Energy/M
- Heavy field does not propagate any more:
  Local (pointlike) interactions
- Correct include in the effective theory systematically additional terms

$$\delta\mathcal{L}_{\text{effth}} = \text{Full Theory Result} - \text{Effective Theory Result}$$

- Key Question: Does this work also for loops?
  Loop integration runs over all momenta including UV
- Look at the Two-Point Function of ϕ



Expansion in the external momentum and calculate $\delta\mathcal{L}_{\text{effth}}$

- Mass Renormalization:

$$\delta m = \frac{1}{16\pi^2}\lambda_1^2\left(1+\frac{m^2}{M^2}\right)+\frac{\lambda_2}{32\pi^2}M^2$$

- Renormalization of the kinetic term:

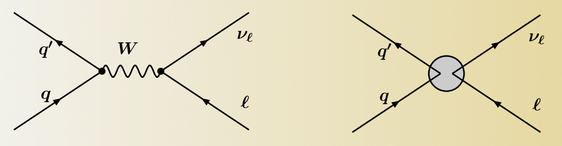
$$\delta Z_\phi = \frac{1}{16\pi^2}\frac{\lambda_1^2}{M^2}$$

- Expansion of the loop diagrams in powers of $1/M$:
  Obtain again higher dimensional operators with perturbatively calculable Wilson coefficiens
- Look at higher n point functions:
  Expand again in powers of $1/M$
- Perturbative construction of $\mathcal{L}_{\mathrm{effth}}$

... now, let's look at real stuff

Effective Weak Hamiltonian

- Start out from the Standard Model
- $W^\pm$, Z^0, top: much heavier than any hadron mass
- "integrate out" these particles at the scale $\mu \sim M_{\rm Hadron}$



- W has zero range in this limit:

$$\langle 0 | T[W^*_\mu(x) W_\nu(y)] | 0 \rangle \to g_{\mu\nu} \frac{1}{M_W^2} \delta^4(x-y)$$

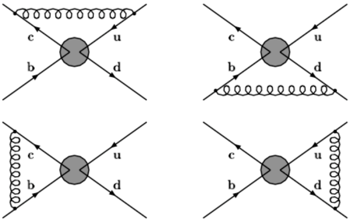
$H_{\rm eff}$ for b decays at scales below M_W

- Effective interaction ($\mathcal{L}_{\rm efft} = -H_{\rm eff}$) :

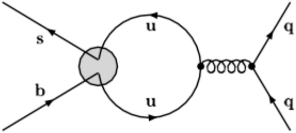
$$H_{\rm eff} = \frac{4G_F}{\sqrt{2}} \lambda_{\rm CKM} \sum_k \hat{C}_k(\Lambda)\mathcal{O}_k(\Lambda) \quad G_F = \frac{g^2}{4\sqrt{2}M_W^2}$$

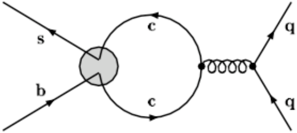
- "Tree" Operators"

$$\begin{aligned} \mathcal{O}_1 &= (\bar{c}_{L,i}\gamma_\mu s_{L,j})\,(\bar{d}_{L,j}\gamma_\mu u_{L,i}) \quad , \\ \mathcal{O}_2 &= (\bar{c}_{L,i}\gamma_\mu s_{L,i})\,(\bar{d}_{L,j}\gamma_\mu u_{L,j}) \quad . \end{aligned}$$



- If two flavours are equal: QCD Penguin Operators





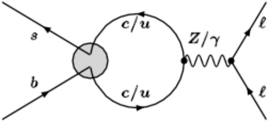
$$\mathcal{O}_3 = (\bar{s}_{\mathrm{L},i}\gamma_\mu b_{\mathrm{L},i}) \sum_{q=u,d,s,c,b} (\bar{q}_{\mathrm{L},j}\gamma^\mu q_{\mathrm{L},j}) \,,$$

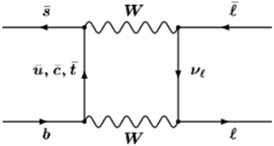
$$\mathcal{O}_4 = (\bar{s}_{\mathrm{L},i}\gamma_\mu b_{\mathrm{L},j}) \sum_{q=u,d,s,c,b} (\bar{q}_{\mathrm{L},j}\gamma^\mu q_{\mathrm{L},i}) \,,$$

$$\mathcal{O}_5 = (\bar{s}_{\mathrm{L},i}\gamma_\mu b_{\mathrm{L},i}) \sum_{q=u,d,s,c,b} (\bar{q}_{\mathrm{R},j}\gamma^\mu q_{\mathrm{R},j}) \,,$$

$$\mathcal{O}_6 = (\bar{s}_{\mathrm{L},i}\gamma_\mu b_{\mathrm{L},j}) \sum_{q=u,d,s,c,b} (\bar{q}_{\mathrm{R},j}\gamma^\mu q_{\mathrm{R},i}) \,.$$

- Electroweak Penguins:
  Replace the Gluon by a Z_0 or Photon: $\mathcal{P}_7 \cdots \mathcal{P}_{10}$
- Rare (FCNC) Processes:





$$\mathcal{O}_7 = \frac{e}{16\pi^2} m_b (\bar{s}_{\mathrm{L},\alpha} \sigma_{\mu\nu} b_{\mathrm{R},\alpha}) F^{\mu\nu}$$

$$\mathcal{O}_8 = \frac{g}{16\pi^2} m_b (\bar{s}_{\mathrm{L},\alpha} T^a_{\alpha\beta} \sigma_{\mu\nu} b_{\mathrm{R},\alpha}) G^{a\mu\nu}$$

$$\mathcal{O}_9 = \frac{1}{2} (\bar{s}_{\mathrm{L}} \gamma_\mu b_{\mathrm{L}}) (\bar{\ell} \gamma^\mu \ell)$$

$$\mathcal{O}_{10} = \frac{1}{2} (\bar{s}_{\mathrm{L}} \gamma_\mu b_{\mathrm{L}}) (\bar{\ell} \gamma^\mu \gamma_5 \ell)$$

Renormalization

For the practitioneer

- QCD is at $\mu \sim M_W$ still perturbative: $\alpha_s(M_W) \ll 1$
- Hadronic matrix elements evaluated at $\mu \sim M_W$ still contain perturbative QCD pieces
- Is there a way to compute these perturbative pieces?
- Consider the QCD renormalization of the operators
- QCD renormalization group

Renormalization Group Running

- H_{eff} is defined at the scale Λ, where we integrated out the particles with mass Λ: General Structure

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \lambda_{\text{CKM}} \sum_k \hat{C}_k(\Lambda) \mathcal{O}_k(\Lambda)$$

- $\mathcal{O}_k(\Lambda)$: The matrix elements of $\mathcal{O}_k$ have to be evaluated (“normalized”) at the scale Λ.
- $\hat{C}_k(\Lambda)$: Short distance contribution, contains the information about scales $\mu > \Lambda$
- Matrix elements of $\mathcal{O}_k(\Lambda)$: Long Distance Contribution, contains the information about scales $\mu < \Lambda$

- We could as well imagine a situation with a different definition of “long” and “short” distances, defined by a scale μ, in which case

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \lambda_{\text{CKM}} \sum_k C_k(\Lambda/\mu) \mathcal{O}_k(\mu)$$

- Key Observation: The matrix elements of H_{eff} are physical Quantities, thus cannot depend on the arbitrary choice of μ

$$0 = \mu \frac{d}{d\mu} H_{\text{eff}}$$

- compute this ....

$$0 = \sum_i \left(\mu \frac{d}{d\mu} C_i(\Lambda/\mu) \right) \mathcal{O}_i(\mu) + C_i(\Lambda/\mu) \left(\mu \frac{d}{d\mu} \mathcal{O}_i(\mu) \right)$$

- Operator Mixing: Change in scale can turns the operator $\mathcal{O}_i$ into a linear combination of operators (of the same dimension)

$$\mu \frac{d}{d\mu} \mathcal{O}_i(\mu) = \sum_j \gamma_{ij}(\mu) \mathcal{O}_j(\mu)$$

and so

$$\sum_i \sum_j \left(\left[\delta_{ij} \mu \frac{d}{d\mu} + \gamma_{ij}(\mu) \right] C_i(\Lambda/\mu) \right) \mathcal{O}_j(\mu) = 0$$

- Assume: The operators $\mathcal{O}_j$ from a basis, then

$$\sum_i \left[\delta_{ij}\mu\frac{d}{d\mu} + \gamma_{ij}^T(\mu)\right] C_j(\Lambda/\mu) = 0$$

- QCD: Coupling constant α_s depends on μ: β-function

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta(\alpha_s(\mu))$$

- C_j depend also on α_s

$$\mu\frac{d}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)$$

- In an appropriate scheme γ_{ij} depend on μ only trough α_s:
$\gamma_{ij}(\mu) = \gamma_{ij}(\alpha_s(\mu))$

- Renormalization Group Equation (RGE) for the coefficients

$$\sum_i \left[\delta_{ij} \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) + \gamma_{ij}^T(\alpha_s) \right] C_j(\Lambda/\mu, \alpha_s) = 0$$

- This is a system of linear differential equations:
  $\rightarrow$ Once the initial conditions are known, the solution is in general unique
- RGE Running: Use the RGE to relate the coefficients at different scales

- The coefficients are at $\mu = \Lambda$ (at the “matching scale”)

$$C_i(\Lambda/\mu = 1, \alpha_s) = \sum_n a_i^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^n$$

perturbative calculation

- Perturbative calculation of the RG functions β and γ_{ij}

$$\beta(\alpha_s) = \alpha_s \sum_{n=0} \beta^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \qquad \gamma_{ij}(\alpha_s) = \sum_{n=0} \gamma_{ij}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

- RG functions can be calculated from loop diagrams:

$$\beta^{(0)} = -\frac{2}{3}(33 - 2n_{\mathrm{f}})$$

γ_{ij} depends on the set of $\mathcal{O}_i$

- Structure of the perturbative expansion of the coefficient at some other scale

$$
\begin{aligned}
& c_i(\Lambda/\mu, \alpha_s) = \\
& b_i^{00} \\
+ \quad & b_i^{11} \left(\frac{\alpha_s}{4\pi}\right) \ln \frac{\Lambda}{\mu} + b_i^{10} \left(\frac{\alpha_s}{4\pi}\right) \\
+ \quad & b_i^{22} \left(\frac{\alpha_s}{4\pi}\right)^2 \ln^2 \frac{\Lambda}{\mu} + b_i^{21} \left(\frac{\alpha_s}{4\pi}\right)^2 \ln \frac{\Lambda}{\mu} + b_i^{20} \left(\frac{\alpha_s}{4\pi}\right)^2 \\
+ \quad & b_i^{33} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln^3 \frac{\Lambda}{\mu} + b_i^{32} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln^2 \frac{\Lambda}{\mu} + b_i^{31} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln \frac{\Lambda}{\mu} + \cdots \quad ,
\end{aligned}
$$

- LLA (Leading Log Approximation):
  Resummation of the b_i^{nn} terms

$$C_i(\Lambda/\mu, \alpha_s) = \sum_{n=0}^{\infty} b_i^{nn} \left(\frac{\alpha_s}{4\pi}\right)^n \ln^n \frac{\Lambda}{\mu}$$

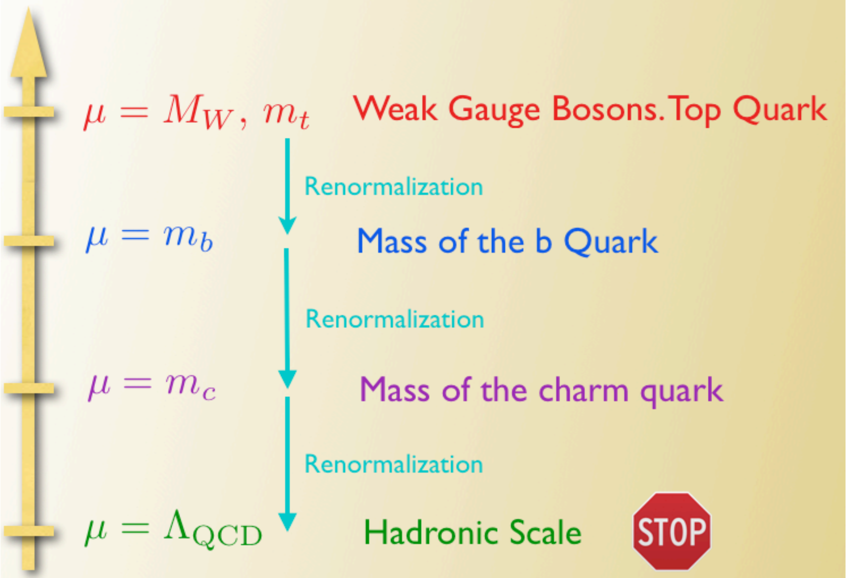
$\rightarrow$ leading terms in the expansion of the RG functions

- NLLA (Next-to-leading log approximation):

$$C_i(\Lambda/\mu, \alpha_s) = \sum_{n=0}^{\infty} \left[b_i^{nn} + b_i^{n+1,n} \left(\frac{\alpha_s}{4\pi}\right)\right] \left(\frac{\alpha_s}{4\pi}\right)^n \ln^n \frac{\Lambda}{\mu}$$

$\rightarrow$ next-to-leading terms of the RG functions

- Typical Proceedure:
  - “Matching” at the scale $\mu = M_W$
  - “Running” to a scale of the order $\mu = m_b$
  - $\to$ includes operator mixing
- Resummation of the large logs $\ln(M_W^2/m_b^2)$
  - “Matching” at the scale $\mu = m_b$
  - “Running” to the scale m_c
- Resummation of the “large” logs $\ln(m_b^2/m_c^2)$
- ...
- Until $\alpha_s(\mu)$ becomes too large ...


$\mu = M_W, m_t$
Weak Gauge Bosons. Top Quark
Renormalization
$\mu = m_b$
Mass of the b Quark
Renormalization
$\mu = m_c$
Mass of the charm quark
Renormalization
$\mu = \Lambda_{QCD}$
Hadronic Scale
STOP

Example: Weak Hamiltonian

- At the "matching scale" M_W

$$C_1(M_W) = 0 \qquad C_2(M_W) = -1$$

- At a smaller scale $\mu \leq M_W$

$$\begin{aligned} C_1(\mu) &= \frac{1}{2} C_2(M_W) \left(\eta^{6/23} - \eta^{-12/23}\right) \\ C_2(\mu) &= \frac{1}{2} C_2(M_W) \left(\eta^{6/23} + \eta^{-12/23}\right) \end{aligned}$$

with

$$\eta = \frac{\alpha_s(M_W)}{\alpha_s(\mu)}$$

- Coefficients of the Operators (One Loop)

$C_i(\mu)$	$\mu = 10.0\,\mathrm{GeV}$	$\mu = 5.0\,\mathrm{GeV}$	$\mu = 2.5\,\mathrm{GeV}$
C_1	0.182	0.275	0.40
C_2	−1.074	−1.121	−1.193
C_3	−0.008	−0.013	−0.019
C_4	0.019	0.028	0.040
C_5	−0.006	−0.008	−0.011
C_6	0.022	0.035	0.055

$C_i(\mu)$	$\mu = 2.5$ GeV	$\mu = 5$ GeV	$\mu = 10$ GeV
C_7^{eff}	−0.334	−0.299	−0.268
C_8^{eff}	−0.157	−0.143	−0.131
$\frac{2\pi}{\alpha} C_9$	1.933	1.788	1.494

Summary Part I

- Effective Theories are good once disparate mass scales appear
- Natrue seems to be a “tower” of effective field theories
- “Matching” to a “full” theory (may be also the effective theory one step higher in the tower
- Renormalization Group “Running” to lower scales μ: Resummation of Logs of the form $\ln(\Lambda^2/\mu^2)$
- Real Application: $\boxed{\Lambda_{NP} \gg M_W \gg m_b \gg m_c \gg \Lambda_{\rm QCD}}$