Flavour physics in the SM: Exercise I

Tobias Hurth, Flavianet school 2010, Bern, 21.6-2.7.2010

1. An example of calculable mixing angles: The properties of the mass matrix can be translated into relations between mass values and mixing angles. Here is an illustrative example of such a model. Consider a simple 2×2 hermitian fermion mass matrix of the form

$$M = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix}. \tag{1}$$

Show that the mixing angle θ whih characterizes the 2×2 unitary matrix which diagonalizes M is related to the mass eigenvalues by

$$\tan\theta = \sqrt{\frac{m_1}{m_2}}. \tag{2}$$

2. CKM Matrix and CP Violation: The fermionic part of the standard model is determined by the mass terms

$$\mathcal{L}_{\rm Yuk} = m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s + m_c \bar{c}c + m_b \bar{b}b + m_t \bar{t}t, \tag{3}$$

and the gauge interactions (neglecting neutral currents and the lepton part)

$$\mathcal{L}_{\rm Vff} = \frac{g}{\sqrt{2}} \bar{u}^i_L \gamma^\mu V_{ij} d^j_L W^+_\mu + h.c. \tag{4}$$

Here V is the unitary CKM matrix.

i) Assume all masses m_q are different from each other. Find all transformations $q_L \to q'_L$, $q_R \to q'_R$ that leave (3) invariant. How do they act on (4), i.e. how do the CKM matrix elements transform? Hint: Express the mass term $m_u \bar{u}u$ in terms their chiral components.

ii) Show that the following quantities are invariant under the transformation found in (i):

- the so called *moduli* of the matrix elements, $|V_{ij}|^2$
- the quartets $Q_{ijkl} = V_{ij} V_{kl} V^*_{il} V^*_{kj}$
- the Jarlskog parameter J defined by (ϵ_{ijk} being the totally antisymmetric tensor)

$$\mathfrak{Im}\left(V_{ij} V_{kl} V^*_{il} V^*_{kj}\right) = J \sum_{m,n=1}^{3} \epsilon_{ikm} \epsilon_{jln}\,. \tag{5}$$

iii) An exact parametrization of V in terms of three angles and a phase was given by Kobayashi and Maskawa:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{6}$$

Here s_{ij} and c_{ij} denote $\sin\theta_{ij}$ and $\cos\theta_{ij}$ respectively. Thus, the four parameters are θ_{12}, θ_{23} and θ_{13} as well as δ. Calculate the Jarlskog parameter in this parametrization. When does J vanish, i.e. what are the conditions on θ_{ij} and δ to have (or not have) CP violation?

iv) Now assume that $m_s = m_b$. Since both quarks have the same charge, they are indistinguishable now and can be unitarily rotated into each other.

(a) Find a rotation

$$\begin{pmatrix} s \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} s' \\ b' \end{pmatrix} = U \begin{pmatrix} s \\ b \end{pmatrix}$$

such that the up-quark does not couple to b'.

(b) In this new basis, the first line of V can now be written as $(\cos\theta, \sin\theta, 0)$. Make an ansatz for the rest of V, using six complex variables. Use unitarity relations to express three of these variables in terms of the others.

(c) Now separate the remaining three parameters into an absolute value and a phase. Use the transformations of exercise (i) to remove all phases from the resulting matrix.

Hints: A possible result from (b) could look like $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -b\tan\theta & b & -fe^*/b^* \\ -e\tan\theta & e & f \end{pmatrix}$, with three complex parameters b, e and f.

v) Consider the quantity

$$C = -iU^\dagger[D_u, V D_d V^\dagger]U, \tag{7}$$

where V is the CKM matrix, D_u and D_d are the diagonal, normalized mass matrices in the up respectively down quark sector, i.e. $D_u = \mathrm{diag}(m_u/m_t, m_c/m_t, 1)$ and similarly$D_d = \mathrm{diag}(m_d/m_b, m_s/m_b, 1)$, and U is a unitary matrix. The brackets denote the commutator. Calculating the determinant of C one finds that

$$\det(C) = -2F_u F_d J, \tag{8}$$
$$F_u = (m_t - m_c)(m_t - m_u)(m_c - m_u)/m_t^3, \tag{9}$$
$$F_d = (m_b - m_s)(m_b - m_d)(m_s - m_d)/m_b^3, \tag{10}$$

with J the Jarlskog parameter. You might want to use a computer to check this, or just believe the result[1]. The conditions for CP violation can be stated as a single equation, namely

$$\det(C) \neq 0\,.$$

Note: A theorem by C. Jarlskog states that if a parametrization exists where V is real, then det C vanishes. Therefore det $C \neq 0$ is a sufficient condition for CP violation in the standard model.

Homework 3. Direct CP Asymmetry in $b \to (s+d)\,\gamma$: To first approximation, the inclusive decay rate of $\bar{B} \to X_s\gamma$ is given by the decay rate $b \to s\gamma$. The matrix element for this decay is given by

$$\mathcal{M} = M_t^{(s)} + M_c^{(s)} + M_u^{(s)}, \tag{11}$$

where the leading contributions come from penguin diagrams with a top-, charm-, or an up-quark running in the loop.

- Show using unitarity of the CKM matrix that the matrix element can be rewritten as

$$\mathcal{M} = \lambda_u^{(s)} A_u^{(s)} + \lambda_c^{(s)} A_c^{(s)} \tag{12}$$

where $\lambda_u^{(s)} = V_{ub}V_{us}^*$, $\lambda_c^{(s)} = V_{cb}V_{cs}^*$.

[1] You can find more about this in the original paper by Jarlskog: Phys.Rev.Lett **55** (1985) 1039

As stated above, to leading order the decay rate can be obtained using Fermi's golden rule:

$$\Gamma\left(\bar{B} \to X_s\gamma\right) \simeq \Gamma\left(b \to s\gamma\right) \propto |\mathcal{M}|^2. \tag{13}$$

- Calculate the direct (unnormalized) CP asymmetry

$$A_{CP}\left(b \to s\gamma\right) = \Gamma\left(b \to s\gamma\right) - \Gamma\left(\bar{b} \to \bar{s}\gamma\right) \tag{14}$$

  and extract the weak phase explicitly.

- Calculate the analogous quantity for the partonic $b \to d\gamma$ decay rate

The "untagged" CP asymmetry is defined as

$$A_{CP}\left(b \to (s+d)\,\gamma\right) = \Gamma\left(b \to s\gamma\right) + \Gamma\left(b \to d\gamma\right) - \Gamma\left(\bar{b} \to \bar{s}\gamma\right) - \Gamma\left(\bar{b} \to \bar{d}\gamma\right) \quad . \tag{15}$$

U-spin symmetry relates the s and the d quark and it is a subgroup of the $SU(3)$ flavour group. If we assume U-spin symmetry with respect to the strong interactions, it follows that the amplitudes A_i in the strange- and down-quark sector are equal.

- Show that in the U-spin limit

$$A_{CP}\left(b \to (s+d)\,\gamma\right) = 0 \quad . \tag{16}$$

- What happens if we assume U-spin symmetry not just with respect to the strong interaction, but for all interactions?

Flavour physics in the SM: Solutions I

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1. An example of calculable mixing angles: The mass matrix can be diagonalized using an orthogonal transformation, e.g.

$$SMS^\dagger = M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \text{or} \quad M = S^\dagger M_d S \tag{1}$$

where

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

From the condition that $M_{11} = 0$ we get

$$S^*_{1i}(M_d)_{ij}S_{j1} = m_1\cos^2\theta + m_2\sin^2\theta = 0, \tag{3}$$

or

$$\tan\theta = \sqrt{\frac{(-1)m_1}{m_2}}. \tag{4}$$

This corrected formula reflects the fact that in this *illustrative (unphysical)* example one mass is negative. However, attempts to relate the Cabibbo angle to the strange and down quark masses have been carried out along such approaches.

Source: T.-P. Cheng, L.-F. Li 'Gauge theory of elementary particle physics: Exercises'

2. CKM Matrix and CP Violation: The fermionic part of the standard model is determined by the mass terms

$$\mathcal{L}_{\text{Yuk}} = m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s + m_c\bar{c}c + m_b\bar{b}b + m_t\bar{t}t, \tag{5}$$

and gauge interactions (neglecting neutral currents and the lepton part)

$$\mathcal{L}_{\text{Vff}} = \frac{g}{\sqrt{2}}\,\bar{u}^i_L\gamma^\mu V_{ij}d^j_L W^+_\mu + h.c. \tag{6}$$

Here V is the unitary CKM matrix.

i) Using the projectors on the chiral components P_L and P_R we can decompose each mass term as follows

$$m_u\bar{u}u = m_u(\bar{u}^\dagger P_L + \bar{u}^\dagger P_R)\gamma^0(P_L u + P_R u) = m_u(\bar{u}_L u_R + \bar{u}_R u_L). \tag{7}$$

The first line of equation (6) reads

$$\frac{g}{\sqrt{2}}\left(\bar{u}_L\gamma^\mu V_{ud}d_L + \bar{u}_L\gamma^\mu V_{us}s_L + \bar{u}_L\gamma^\mu V_{ub}b_L\right)W^+_\mu + \text{h.c.} \tag{8}$$

In order to leave the mass terms (5) invariant, the left and righthanded components of the fields have to transform identically. Furthermore transformations of one type of quark into another are forbidden, since they have different masses. This leaves us with

$$q_{i,L} \longrightarrow q'_{i,L} = e^{i\phi_i}q_{i,L} \qquad q_{i,R} \longrightarrow q'_{i,R} = e^{i\phi_i}q_{i,R}\,. \tag{9}$$

In total these are six transformations with in total six free parameters, one for each quark. We denote with φ_i the phases of the up-type quarks and with ϕ_i the one of the down type quarks. The six parameters are then $(\varphi_u, \varphi_c, \varphi_t, \phi_d, \phi_s, \phi_b)$. From (6) it follows that the CKM matrix elements transform as follows:

$$V_{ij} \longrightarrow e^{i(\varphi_j - \phi_i)} V_{ij}\,, \tag{10}$$

where i is the index of an up-type quark and j the index of a down-type quark.

ii) In the following we show the invariance of the moduli, the quartets and the Jarlskog parameter under the phase shift of (i):

$$|V_{ij}|^2 = V_{ij}^* V_{ij} \longrightarrow V_{ij}^* e^{-i(\phi_j - \phi_i)} e^{i(\phi_j - \phi_i)} V_{ij} = |V_{ij}|^2 \tag{11}$$

$$Q_{ijkl} = V_{ij} V_{kl} V_{il}^* V_{kj}^* \longrightarrow e^{i(\phi_j - \phi_i + \phi_l - \phi_k - \phi_l + \phi_i - \phi_j + \phi_k)} V_{ij} V_{kl} V_{il}^* V_{kj}^* = Q_{ijkl} \tag{12}$$

The Jarlskog parameter is the imaginary part of a quartet Q_{ijkl} and therefore invariant!

iii) In terms of these parameterisation, the Jarlskog parameter is given by

$$J = s_{12} s_{13} s_{23} c_{12} c_{23} c_{13}^2 \sin\delta. \tag{13}$$

The sine and cosine of every angle θ_{ij} appears. In order for J not to vanish, we must have

$$\theta_{ij} \neq 0, \frac{\pi}{2} \quad \forall\, j \qquad \text{and} \qquad \delta \neq 0, \pi\,. \tag{14}$$

iv) (a) We can simply choose s' as

$$s' = V_{us} s + V_{ub} b. \tag{15}$$

Now consider your result from exercise (a), equation (8). Under the above transformation the we observe that

$$(\bar{u}_L \gamma^\mu V_{us} s_L + \bar{u}_L \gamma^\mu V_{ub} b_L) \longrightarrow \bar{u}_L \gamma^\mu V_{us}' s_L' \tag{16}$$

the coupling of the up quark u to b' is simply zero. The actual value of V_{us}' can be derived from unitarity, however this is not relevant here.

(b) The most general form for the first line of V is now $(x, y, 0)$. Unitarity requires that $xx^* + yy^* = 1$, i.e. a solution is given by $(\cos\theta, \sin\theta, 0)$. Possible phases of x and y can be rotated away using appropriate parameters ϕ_d, ϕ_s in the transformations of exercise (i). The most general ansatz for V now is given by

$$V = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ a & x & c \\ d & e & f \end{pmatrix}, \tag{17}$$

with arbitrary complex parameters a, x, c, d, e, f. Using the unitarity relations $V^\dagger V = 1$ and $VV^\dagger = 1$ we can for example derive the following constraints on the parameters:

$$a = -x \tan\theta \tag{18}$$

$$d = -e \tan\theta \tag{19}$$

$$c = -fe^*/x^* \tag{20}$$

This gives us

$$V = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -x\tan\theta & x & -fe^*/x^* \\ -e\tan\theta & e & f \end{pmatrix} \tag{21}$$

(c) Now we separate each parameter into absolute value and phase, e.g. $x = Xe^{i\phi_x}$. We are only interested in the phases, so we drop the absolute values. The matrix then looks like

$$\begin{pmatrix} 1 & 1 & 0 \\ e^{i\phi_x} & e^{i\phi_x} & e^{i(\phi_x+\phi_f-\phi_e)} \\ e^{i\phi_e} & e^{i\phi_e} & e^{i\phi_f} \end{pmatrix}. \tag{22}$$

We now want to use the transformations of exercise (b) to remove as many phases as possible from this matrix. As a first observation, we note that we do not want to reintroduce a phase in the first line. It follows that

$$\phi_u = \phi_d = \phi_s, \tag{23}$$

i.e. these three phases can only be rotated together. We are left with $f$our transformations. Now consider the first two entries in the second line. From exercise (b) we know that these entries transform with $e^{i(\phi_d-\phi_c)}$ and $e^{i(\phi_s-\phi_c)}$ respectively. To remove the phase in V the transformations therefore have to satisfy

$$\phi_d - \phi_c = -\phi_x \qquad \phi_s - \phi_c = -\phi_x. \tag{24}$$

Since $\phi_d = \phi_s$, these equations can be satisfied by adjusting $\phi_d = \phi_s = \phi_c - \phi_x$, keeping ϕ_c arbitrary. We are left with $t$hree transformations.
Now consider the first two entries in the third line. The relevant equations now are

$$\phi_d - \phi_t = -\phi_e \qquad \phi_s - \phi_t = -\phi_e. \tag{25}$$

Now ϕ_d, ϕ_s are already determined, so we have to use ϕ_t. Again since $\phi_d = \phi_s$, both equations are satisfied by $\phi_t = \phi_d + \phi_e$, removing both phases. We are left with $t$wo transformations.
The equations for the remaining two entries read

$$\phi_b - \phi_t = -\phi_f \qquad \phi_b - \phi_c = -(\phi_x + \phi_f - \phi_e). \tag{26}$$

The first equation can be solved by $\phi_b = \phi_t - \phi_f$. This fixes one more transformation, leaving free ϕ_c which can now be chosen to satisfy the last equation, thereby removing all phases from V.
The same chain of arguments can be used to show that V is real if any two equally charged quarks happen to have the same mass. In order to have CP violation, no equally charged quarks may have the same mass!

v) The calculation of det C can be left for the computer. What you should observe here, is the following:
The value of det C is a product of the differences of equally-charged quark masses times the Jarlskog parameter. We already know that CP violation is not possible if two or more equally charged quarks have the same mass. Furthermore CP violation is also proportional to the Jarlskog parameter and vanishes for $J = 0$. Thus the condition det $C \neq 0$ summarizes all necessary conditions to have CP violation.

However there is more. C. Jarlskog has proven a theorem stating that det C vanishes if V is real. Now since det C is parametrization independent, it vanishes also if for a given V there exists a real parametrization V'. It follows that if and only if det $C \neq 0$ for a given V then there exists no transformation that makes V real while leaving invariant the rest of the Lagrangian.
This shows that det $C \neq 0$ is a necessary **and** sufficient condition to have CP violation via the CKM mechanism in the standard model.

Note: This argument breaks down if there is a fourth quark family, and in general can only be used if there is an odd number of quark families.

More information about the Jarlskog theorem can be found on the original paper: Phys. Rev. Lett **55** (1985) 1039.

Homework 3. Direct CP Asymmetry in $b \to (s+d)\,\gamma$:

- We find, that for each fermion running in the loop, the corresponding amplitude is proportional to

$$\mathcal{M}_i \propto V_{bi} V_{is}^* A_i \quad \text{where}\, i = u, c, t \tag{27}$$

We now have, using one of the relations from $V_{CKM}^\dagger VCKM = 1$, that

$$V_{us} V_{ub}^* + V_{tb} V_{ts}^* + V_{cs} V cb^* = 0 \tag{28}$$

So we find for the matrix element

$$\begin{aligned} \mathcal{M}^{(b)} &= M_t^{(S)} + M_c^{(S)} + M_u^{(S)} \\ &= A_t V_{bt} V_{ts}^* + A_c V_{cb} V_{cs}^* + A_t V_{ub} V_{us}^* \\ &= V_{cb} V_{cs}^* (A_c - A_t) + V_{ub} V_{us}^* (A_u - A_t) \\ &= \lambda_u^{(s)} A_u^{(s)} + \lambda_c^{(s)} A_c^{(s)}. \end{aligned} \tag{29}$$

Playing the same game for the decay $\bar{b} \to \bar{s}\gamma$, we find

$$\mathcal{M}^{(\bar{b})} = \lambda_u^{(s)*} A_u^{(s)} + \lambda_c^{(s)*} A_c^{(s)}. \tag{30}$$

- Therefore, we find for the unnormalized asymmetry

$$\begin{aligned} A_{CP}(b \to s\gamma) &= \Gamma(b \to s\gamma) - \Gamma(\bar{b} \to \bar{s}\gamma) \\ &= \left| \lambda_u^{(s)} A_u^{(s)} + \lambda_c^{(s)} A_c^{(s)} \right|^2 - \left| \lambda_u^{(s)*} A_u^{(s)} + \lambda_c^{(s)*} A_c^{(s)} \right|^2 \\ &= 4\Im(A_u^{(s)} A_c^{(s)*}) \Im(\lambda_u^{(s)} \lambda_c^{(s)*}) \\ &= 4\Im(A_u^{(s)} A_c^{(s)*}) \Im(V_{cb} V_{us} V_{cs}^* V_{ub}^*) \\ &= 4\Im(A_u^{(s)} A_c^{(s)*}) J. \end{aligned} \tag{31}$$

The relation

$$\Im(V_{ij} V_{kl} V_{il}^* V_{kj}^*) = J \sum_{m,n} \epsilon_{ikm} \epsilon_{jln} \tag{32}$$

was used and J is given in terms of the standard CKM-matrix parametrization in equation (13).

- Proceeding in a completely analogous way we obtain

$$A_{CP}(b \to d\gamma) = -4\Im(A_u^{(d)} A_c^{(d)*}) J. \tag{33}$$

- Therefore, we obtain for the untagged CP-Asymmetry

$$A_{CP}(b \to (s+d)\gamma) = A_{CP}(b \to d\gamma) + A_{CP}(b \to s\gamma) = 4J\Im(A_u^{(s)} A_c^{(s)*} - A_u^{(d)} A_c^{(d)*}). \tag{34}$$

It is now easy to see that assuming U-spin symmetry, and therefore that $A_q^{(s)} = A_q^{(d)}$ for all quarks q, that the untagged CP-Asymmetry vanishes.

- Assuming U-spin symmetry for all interactions would specifically imply that $m_s = m_d$. Recalling the result mentioned in class, namely that CP-violation vanishes if there are degenerate quark masses, we immediately derive that the CP-Asymmetry and Jarlskog parameter vanish.

Flavour physics in the SM: Exercise II

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1. Gluinos as Majorana fermions: Within the minimal supersymmetric standard model (MSSM), there are fermionic superpartners of the SM gluon fields. These so-called gluinos are Majorana fermions but as superpartners of the guons they also carry color charge. Why is this possible?

2. Majorana masses for the neutrinos on the tree level via a Higgs-triplet: The lepton fields and Higgs scalar have the following $SU(2)_L \times U(1)$ transfomation properties where the bold number represents the dimensionality of the $SU(2))_L$ representation and the other the value of the $U(1)$ corresponding to the hypercharge $Y = Q - T_3$:

$$L_L = \begin{pmatrix} \nu_L \\ l_L^- \end{pmatrix}_L \sim (\mathbf{2}, -\tfrac{1}{2}) \quad L_R \sim (\mathbf{1}, -1) \tag{1}$$

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \sim (\mathbf{2}, \tfrac{1}{2}) \tag{2}$$

The upper index indicates the electric charge of the component field. There are the following Lorentz invariant terms bilinear in the lepton fields:

$$\begin{array}{rcl} \overline{L}_L L_R & \sim & (\mathbf{2}, \frac{1}{2}) \otimes (\mathbf{1}, -1) \\ \overline{(L_L)^C} L_L & \sim & (\mathbf{2}, -\frac{1}{2}) \otimes (\mathbf{2}, -\frac{1}{2}) \\ \overline{(L_R)^C} L_R & \sim & (\mathbf{1}, -1) \otimes (\mathbf{1}, -1). \end{array} \tag{3}$$

Remember that $\psi^C = C\gamma_0\psi^*$ and $\overline{\psi^C} = \psi^T C$. Note that chiral projection and conjugation do not commute: $(\psi^C)_L = (\psi_R)^C$

- Study the tensor structure regarding $SU(2)_L \times U(1)$ of these terms by performing a Clebsch-Gordan decomposition.
- Then find the multiplets of scalar fields that you can couple in a gauge-invariant way to these terms.

Let us study one possibility further, namely adding a Higgs triplet

$$\boldsymbol{\tau} \cdot \boldsymbol{H} = \tau_1 H_1 + \tau_2 H_2 + \tau_3 H_3 = \begin{pmatrix} H^+ & \sqrt{2}H^{++} \\ \sqrt{2}H^0 & -H^+ \end{pmatrix}, \tag{4}$$

with τ the Pauli matrices. The Higgs triplet belongs to the adjoint representation transforming under SU(2) like

$$\boldsymbol{\tau} \cdot \boldsymbol{H} \to e^{i\boldsymbol{\theta}\cdot\boldsymbol{\tau}} \boldsymbol{\tau} \cdot \boldsymbol{H} e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\tau}}. \tag{5}$$

This gives rise to the additional Yukawa coupling and to the additional scalar coupling:

$$f\overline{(L_{iL})^C} L_{jL} \boldsymbol{H} (\epsilon\boldsymbol{\tau})_{ij} + \mu\phi_i\phi_j \boldsymbol{H}^* (\epsilon\boldsymbol{\tau})_{ij} + h.c.; \tag{6}$$

where $\epsilon = i\tau_2$. Note that ϵ_{ij} and $(\epsilon\boldsymbol{\tau})_{ij}$ are antisymmetric and symmetric, respectively.

- Convince yourself of the charges of the field components by writing down their quantum numbers and calculating the electric charge (Remember: $Y = Q - T_3$).
- Give the neutral component of the Higgs triplet a vacuum expectation value v_H to obtain a Majorana mass term for the neutrino field.

Flavour physics in the SM: Solutions II

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1. Gluinos as Majorana fermions: The gluinos like the gluons transform under the adjoint representation which is real.

2. Neutrino masses via a Higgs-triplet

- We can write the following terms that are bilinear in lepton fields and make a Clebsch-Gordan-decomposition:

$$\begin{aligned} \overline{L}_L L_R &\sim (\mathbf{2}, \tfrac{1}{2}) \otimes (\mathbf{1}, -1) = (\mathbf{2}, -\tfrac{1}{2}) \\ \overline{(L_L)^C} L_L &\sim (\mathbf{2}, -\tfrac{1}{2}) \otimes (\mathbf{2}, -\tfrac{1}{2}) = (\mathbf{1}, -1) \oplus (\mathbf{3}, -1) \\ \overline{(L_R)^C} L_R &\sim (\mathbf{1}, -1) \otimes (\mathbf{1}, -1) = (\mathbf{1}, -2). \end{aligned} \tag{1}$$

  The hypercharge charge coresponds to a $U(1)$ and can therefore just be added. The 2-dimensional $SU(2)_L$ representation is equivalent to the D_j spin - representation with $j = \frac{1}{2}$ from quantum mechanics. We can therefore use

$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \ldots + D_{|_1-j_2|} \tag{2}$$

  to decompose the tensor product as follows (just adding the resp. hypercharge quantum numbers):

$$\begin{aligned} (\mathbf{2}, \tfrac{1}{2}) \otimes (\mathbf{1}, -1) &\sim D_{\frac{1}{2}} \otimes D_0 = D_{\frac{1}{2}} \sim (\mathbf{2}, -\tfrac{1}{2}) \\ (\mathbf{2}, -\tfrac{1}{2}) \otimes (\mathbf{2}, -\tfrac{1}{2}) &\sim D_{\frac{1}{2}} \otimes D_{\frac{1}{2}} = D_1 \oplus D_0 \sim (\mathbf{1}, -1) \oplus (\mathbf{3}, -1). \end{aligned} \tag{3}$$

- If we have the SM-Higgs as above, only the Yukawa couplings, $\bar{L}_R L_L \Phi + h.c.$ are present.

  But we could also add the following terms: a charged singlet $h^+ \sim (\mathbf{1}, 1)$, a doubly charged singlet $R^{++} \sim (\mathbf{1}, 2)$ or a triplet: $\boldsymbol{H} \sim (\mathbf{3}, 1)$.

- We want the mass term to be symmetric. Therefore, we have to somehow create a symmetric mass matrix out of $\tau \cdot \boldsymbol{H}$. We do this by introducing

$$\epsilon = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4}$$

  the 2-index equivalent to the Levi-Civita tensor.

- In analogy to angular momentum in quantum mechanics, we have that the Higgs triplet with three components H_-, H_0, H_+ with $T^3 = -1, 0, 1$. Furthermore, we know that each complex field H_i has the hypercharge $Y = 1$. Since $Q = T^3 + Y$ we obtain for the respective charges $Q = 0, +1, +2$.

  We can also write down the fields and calculate the T^3 component of each piece of the triplet by the commutator with τ_3. We have

$$\boldsymbol{\tau} \cdot \boldsymbol{H} = \begin{pmatrix} H_3 & H_1 - iH_2 \\ H_1 + iH_2 & -H_3 \end{pmatrix} = \begin{pmatrix} H_3 & \sqrt{2}H_+ \\ \sqrt{2}H_- & -H_3 \end{pmatrix}. \tag{5}$$

  The commutator leads to

$$[\frac{\tau_3}{2}, \boldsymbol{\tau} \cdot \boldsymbol{H}] = \begin{pmatrix} 0 \times H_3 & 1 \times \sqrt{2}H_+ \\ (-1) \times \sqrt{2}H_- & 0 \times -H_3 \end{pmatrix} \tag{6}$$

  from which we can read off T^3.